

SEMI-HARMONICITY, INTEGRAL MEANS AND EULER TYPE VECTOR FIELDS

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ABSTRACT. The Dirichlet product of functions on a semi-Riemann domain and generalized Euler vector fields, which include the radial, $\bar{\partial}$ -Euler, and the $\bar{\partial}$ -Neumann vector fields, are introduced. The integral means and the harmonic residues of functions on a Riemann domain are studied. The notion of semi-harmonicity of functions on a complex space is introduced. It is shown that, on a Riemann domain, the semi-harmonicity of a locally forwardly L^2 -function is characterized by local mean-value properties as well as by weak-harmonicity. In particular, the Weyl's Lemma is extended to a Riemann domain.

1. INTRODUCTION

¹As higher dimensional analogues of Riemann surfaces, the Riemann domains played a fundamental role in the early development of function theory of several complex variables (see [4], pp. 12-13). Such a space is given by a complex manifold M together with a holomorphic map p of M into a domain in \mathbb{C}^m such that each fiber of p is discrete. This allows for the consideration of m -dimensional domains that do not lie within \mathbb{C}^m .

It is well-known that on a Euclidean space harmonic functions are characterized by their mean-value properties over balls and spheres. In view of recent interest in non-smooth domains in analysis, it seems natural to consider similar properties for functions defined on a *semi-Riemann domain*, where singular or branch points as well as some non-discrete fibers may exist, thus allowing for possibly non-Stein parabolic spaces lying over a domain in \mathbb{C}^m . In this paper *semi-harmonic functions* on a complex space are introduced. It is shown that for a continuous function on a Riemann domain, the "semi-harmonicity" is characterizable in terms of the local behaviour of the function such as the *solid*, *spherical*, as well as by the *near*, resp. *weak*, *harmonicity*. Furthermore, the Weyl's Lemma can be extended ²: every

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²The implication "(1) \Rightarrow (2)" given in Theorem 4.2 of the above quoted paper is not correct as it stands unless under some further conditions on ϕ for instance, if either $\phi \in C^0(X)$ or ϕ admits locally branchwise L^1 -direct image $\hat{\phi}^j \in L^1_{\text{loc}}(U')$ at each point of X , for each branch of U (under such circumstances the original proof remains valid).

locally forwardly L^2 weakly harmonic or semi-harmonic function on a Riemann domain is induced by a semi-harmonic function (Corollary 4.1).

For later applications to local and global characterizations of semi-harmonicity and holomorphicity of functions on a normal semi-Riemann domain (see [15]), a class of generalized Euler vector fields is introduced (see §2 & §5). The point of interest here lies in the fact that the Cauchy-Riemann, the $\bar{\partial}$ -Euler, the $\bar{\partial}$ -Neumann as well as the *radial* vector fields can be globally defined from a unified viewpoint. The relation between semi-harmonicity, *Dirichlet product*, and *harmonic residues* is also studied. Integral representation of the Bochner-Martinelli type and applications will be considered in a subsequent paper [15].

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2. PRELIMINARIES

In what follows every complex space is assumed to be reduced and has a countable topology. The notions of C^k -differential forms, the exterior differentiation d , the operators ∂ , $\bar{\partial}$ and $d^c := (1/4\pi i)(\partial - \bar{\partial})$ are well-defined on complex spaces despite the presence of singularities (see [14], Chap. 4). Let X be a complex space of dimension $m > 0$ and $D \subseteq X$ an open subset. Denote by $C^\mu(D)$ the set of all \mathbb{C} -valued functions of class C^μ (when $\mu = \beta$, locally bounded functions) on D , and by $A^{k,\mu}(D)$ the set of \mathbb{C} -valued k -forms of class C^μ (when $\mu = \lambda$, \mathbb{C} -valued locally Lipschitz k -forms ([14], §4)) on D . The sets $C^\mu(\bar{D})$ and $A^{k,\lambda}(\bar{D})$ are similarly defined.

Denote by $\|z\|$ the Euclidean norm of $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, where $z_j = x_j + i y_j$. Let the space \mathbb{C}^m be oriented so that the form $v^m := (dd^c \|z\|^2)^m$ is positive. Let $p : X \rightarrow \mathbb{C}^m$ be a holomorphic map. Set $p^{[a]} := p - p(a)$, $\forall a \in X$. Clearly the form

$$(2.1) \quad v_p := dd^c \|p^{[a]}\|^2 = \left(\frac{i}{2\pi}\right) \partial \bar{\partial} \|p^{[a]}\|^2$$

is non-negative and independent of a . Observe that the function $\|p^{[a]}\|$ satisfies the Monge-Ampère equation:

$$(2.2) \quad (dd^c \log \|p^{[a]}\|^2)^m = 0.$$

Therefore the form

$$(2.3) \quad \sigma_a := \frac{1}{\|p^{[a]}\|^{2m}} d^c \|p^{[a]}\|^2 \wedge v_p^{m-1}$$

is d -closed.

Let dv_r , respectively, $d\sigma_r$, denote the Euclidean volume element of the r -ball $\mathbb{B}(r)$, respectively, the $(2m-1)$ -sphere $\mathbb{S}(r) = \mathbb{S}^{2m-1}(r)$, in \mathbb{C}^m . Set $dv_{[a],r} := (p^{[a]})^* dv_r$, $d\sigma_{[a],r} := (p^{[a]})^* d\sigma_r$. In particular, denoting by $t_{(-a')}$ the translation: $z \mapsto z - a'$ (where $a' \in \mathbb{C}^m$), the forms $dv_{[a'],r} := t_{(-a')}^* dv_r$, $d\sigma_{[a'],r} := t_{(-a')}^* d\sigma_r$ are defined. For $a \in D$ and $r > 0$, set $D_{[a]}(r) := \{z \in D \mid \|p^{[a]}(z)\| < r\}$, $D_{[a]}[r] := \{z \in D \mid \|p^{[a]}(z)\| \leq r\}$, and $D(r) := \{z \in D \mid \|p(z)\| < r\}$. If p = the identity

map of \mathbb{C}^m , write $\mathbb{B}_{[a']}(r) = D_{[a']}(r)$, $\mathbb{S}_{[a']}(r) = \partial\mathbb{B}_{[a']}(r)$, and omit the subscript if $a' = 0$. Set $|\mathbb{S}| = \text{vol}(\mathbb{S}(1))$ and $|\mathbb{B}| = \text{vol}(\mathbb{B}(1))$.

A complex space X together with a holomorphic map $p : X \rightarrow \Omega$, where Ω is a domain in \mathbb{C}^m , is called a *semi-Riemann domain* (over Ω) iff there exists a thin analytic set Σ in Ω for which the inverse image $\Sigma_p := p^{-1}(\Sigma)$ is thin in X and the restriction $p : X^0 := X \setminus \Sigma_p \rightarrow \Omega_0 := \Omega \setminus \Sigma$ has discrete fibers. If $\Sigma = \emptyset$, then (X, p) is a *Riemann domain* in the sense of [4], p. 19 and [11], p. 135, (also [9], p. 116, where X is assumed a normal space). If $p : X \rightarrow \Omega$ is in addition a local homeomorphism, then (X, p) is said to be *unramified*. Every proper holomorphic map of a pure m -dimensional complex space into a domain $\Omega \subseteq \mathbb{C}^m$ of strict rank m is a semi-Riemann domain ([1], p. 117).

If Y is an irreducible and locally irreducible space and $f : X \rightarrow Y$ is a light, proper, holomorphic map, then by the Andreotti-Stoll's theorem ([13], Lemma 2.3; [1], Lemma 2.2), the map $f : X \rightarrow Y$ is an analytic covering with sheet number $s = \deg(f)$ given by

$$(2.4) \quad \deg(f) := \sum \{\nu_f^y(z) \mid z \in f^{-1}(y)\}, \quad \forall y \in Y.$$

where $\nu_f^y(z)$ denotes the *multiplicity of f at z* ([13], p. 22).

Unless otherwise mentioned, let $p = (p_1, \dots, p_m) : X \rightarrow \Omega$ be a semi-Riemann domain of dimension $m > 0$. For each $a \in D^0 := D \cap X^0$, there exists an open neighborhood N with closure in D^0 such that: i) $p^{-1}(y) \cap \overline{N} = \{a\}$, where $a' := p(a)$; ii) for a sufficiently small ball $U' = \mathbb{B}_{[a']}(\rho)$ in \mathbb{C}^m , $U_a := p^{-1}(U') \cap N = p^{-1}(U') \cap \overline{N}$ is connected and the mapping $p|_{U_a} : U_a \rightarrow U'$ is an analytic covering; iii) every branch V^k , $k = 1, \dots, s_a$, of U_a contains a ; and iv)

$$(2.5) \quad s_a = \deg(p|_{U_a}) = \nu_p^{a'}(a)$$

([13], Proposition 1.3). For convenience call such U_a a *pseudo-ball (of radius ρ) at a* . Let X^* be the largest open subset of X on which p is locally biholomorphic, and set $D^* := D \cap X^*$.

Let $W \subseteq X^*$ be an open set and $T_a(W)$ the real tangent space at a point $a \in W$. Denote by (\cdot, \cdot) the Euclidean inner product (induced under p) on the tangent space $T_a(W)$, $a \in W$. It extends naturally to a bilinear form on the complexified tangent space $\mathbb{C}T_a(W)$. The gradient vector field $\nabla\phi$ of a C^1 -function $\phi : W \rightarrow \mathbb{C}$ is then well-defined in the standard fashion; thus setting $\tilde{x}_j = \text{Re}(p_j)$, $\tilde{y}_j = \text{Im}(p_j)$, $1 \leq j \leq m$, the partial derivatives

$$\phi_{\tilde{x}_j} = \frac{\partial\phi}{\partial\tilde{x}_j} := (\nabla\phi, \frac{\partial}{\partial\tilde{x}_j}); \quad \phi_{\tilde{y}_j} = \frac{\partial\phi}{\partial\tilde{y}_j} := (\nabla\phi, \frac{\partial}{\partial\tilde{y}_j}),$$

are well-defined. The *Cauchy-Riemann*, respectively *anti-Cauchy-Riemann*, *vector fields*, are given by

$$\bar{\partial}_k = \frac{\partial}{\partial\bar{p}_k} := \frac{1}{2} \nabla p_k, \quad \text{respectively,} \quad \partial_k = \frac{\partial}{\partial p_k} := \frac{1}{2} \nabla \bar{p}_k,$$

in X^* for $k = 1, \dots, m$. More generally, if $g \in C^1(D)$ define the associated ∂ -, respectively, $\bar{\partial}$ -, *Euler type vector field*

$$\begin{aligned}
(2.6) \quad \mathcal{E}_g &= 2 \sum_{k=1}^m g_{\bar{p}_k} \partial_k, \\
\bar{\mathcal{E}}_g &= 2 \sum_{k=1}^m g_{p_k} \bar{\partial}_k,
\end{aligned}$$

in D^* , where $g_{\bar{p}_k} := \bar{\partial}_k g$, $g_{p_k} := \partial_k g$. Observe that if $h \in \mathcal{O}(D)$ (resp. $h \in \overline{\mathcal{O}}(D)$), the set of all holomorphic (resp. anti-holomorphic) functions, then $\nabla h = \bar{\mathcal{E}}_h$ (resp. $\nabla h = \mathcal{E}_h$) is an Euler type vector field. To each continuous mapping $\xi = (\xi_1, \dots, \xi_{2m}) : W \rightarrow \mathbb{C}^{2m}$ is associated a (complex) vector field

$$\partial_\xi := \sum_{k=1}^m (\xi_{2k-1} \frac{\partial}{\partial \tilde{x}_k} + \xi_{2k} \frac{\partial}{\partial \tilde{y}_k}),$$

which, for notational convenience, shall be identified with ξ . It follows that

$$(2.7) \quad \partial_{\nabla g} = \mathcal{E}_g + \bar{\mathcal{E}}_g = \sum_{k=1}^m \nabla_k g,$$

where

$$(2.8) \quad \nabla_k g := 2(g_{p_k} \bar{\partial}_k + g_{\bar{p}_k} \partial_k)$$

is the k -th *partial gradient vector field* of g . The following Lemma, which gathers some useful identities, is easily established:

Lemma 2.1. (1) If $\phi, \psi \in C^1(D^*)$, then

$$(2.9) \quad d\phi \wedge d^c \psi \wedge v_p^{m-1} = \frac{1}{4m} \partial_{\nabla \phi}(\psi) v_p^m.$$

(2) If $g \in C^2(D^*)$, then for each pseudo-ball $U \subseteq D^*$, (i)

$$(2.10) \quad dd^c(g v_p^{m-1}) = \frac{1}{4m} (\Delta_{p_U} g) v_p^m,$$

where Δ_{p_U} denotes the p_U -pull-back of the Laplace operator of the Euclidean metric on \mathbb{C}^m ; (ii)

$$(2.11) \quad \partial_{\nabla g}(\phi) = \frac{1}{2} (\Delta_{p_U}(g\phi) - g\Delta_{p_U}\phi - \phi\Delta_{p_U}g), \quad \forall \phi \in C^2(U).$$

3. INTEGRAL AVERAGES

Denote by dD the (maximal) *boundary manifold* of $\mathcal{R}(D)$ in $\mathcal{R}(X)$, the manifold of simple points of X , oriented to the exterior of $\mathcal{R}(D)$ ([14], p. 218). If \overline{U} is compact, $a \in U^0$, and $\phi \in C^0(U_{[a]}[r_0])$, define the *solid*, resp. *spherical*, *mean-value function* of ϕ (with resp. to $p^{[a]}$) by

$$(3.1) \quad \langle \phi \rangle U_{a,r} := \frac{1}{r^{2m}} \int_{U_{[a]}(r)} \phi v_p^m, \quad \forall r \in (0, r_0),$$

$$(3.2) \quad [\phi]U]_{a,r} := \int_{dU_{[a]}(r)} \phi \sigma_a, \quad \forall r \in (0, r_0).$$

Let $\phi \in C^0(D)$ and $a \in D^0$. Then: ϕ is said to have (1) the (local) *solid mean-value property at a* iff there exists a pseudo-ball $U \subseteq D$ at a of radius $r_0 > 0$ such that

$$(3.3) \quad \langle \phi]U \rangle_{a,r} = \nu_p(a) \phi(a) \quad \forall r \in (0, r_0).$$

(2) the (local) *spherical mean-value property at a* iff there exists a pseudo-ball $U \subseteq D$ at a of radius $r_0 > 0$ such that

$$(3.4) \quad [\phi]U]_{a,r} = \nu_p(a) \phi(a), \quad \forall r \in (0, r_0).$$

Given a pseudoball U in X , choose a C^∞ -partition of unity $\{(U_\nu^j, \rho_\nu^j)\}$ in terms of the local (covering) sheets $U_\nu^j \subseteq U_*$ contained in a branch V^j . Given $\phi \in L_{\text{loc}}^1(U)$ (respectively, $C^0(U)$), set $\phi_\nu^j := \rho_\nu^j \phi$ for each pair (j, ν) ; there exists an induced function $\hat{\phi}_\nu^j \in L_{\text{loc}}^1((U_\nu^j)')$ (respectively, $C^0((U_\nu^j)')$) such that $p^* \hat{\phi}_\nu^j = \phi_\nu^j$ on U_ν^j . The following Lemma shows that the local solid and the spherical mean-value properties of ϕ are equivalent:

Lemma 3.1. *Let U be a pseudo-ball at a point $a \in X^0$ of radius r_0 . Then for each $\phi \in C^0(\overline{U})$, the following are equivalent:*

- (a) $\langle \phi]U \rangle_{a,r} = \text{const.} = A, \forall r \in (0, r_0)$;
- (b) $[\phi]U]_{a,r} = \text{const.} = A, \forall r \in (0, r_0)$.

Proof. Observe that, if $\hat{\phi} \in C^0(\mathbb{B}_{[a']}[r_0])$, then

$$(3.5) \quad \int_{\mathbb{B}_{[a']}(r)} \hat{\phi} dv_{[a'],r} = \int_0^r \left(\int_{\mathbb{S}_{[a']}(t)} \hat{\phi} d\sigma_{[a'],t} \right) dt, \quad 0 < r < r_0.$$

Assume at first that $[\phi]U]_{a,t} = A, \forall t \in (0, r_0)$. Denoting by s_a the number of irreducible branches of U , one has for such t ,

$$\begin{aligned} A |\mathbb{S}| t^{2m-1} &= \int_{dU_{[a]}(t) \setminus T} \phi d\sigma_{[a],t} = \sum_{k=1}^{s_a} \sum_{\nu}' \int_{dV_{[a]}^k(t) \cap U_\nu^k \setminus T} \phi_\nu^k d\sigma_{[a],t} \\ &= \sum_{k=1}^{s_a} \sum_{\nu}' \int_{\mathbb{S}_{[a']}(t) \cap (U_\nu^k)'} \hat{\phi}_\nu^k d\sigma_{[a'],t}, \end{aligned}$$

where \sum_{ν}' denotes the limit of a sum of integrals (or functions) over the indices ν of an (increasing) covering of $K_n \cap V^k$ by the U_ν^k , $\{K_n\}$ being a (strictly increasing) exhausting sequence of compact subsets of U^* . Thus, integrating the above relation over $(0, r)$, $0 < r < r_0$, yields

$$\frac{r^{2m}}{2m} |\mathbb{S}| A = \sum_{k=1}^{\mathfrak{s}_a} \sum'_{\nu} \int_{\mathbb{B}_{[a']}(r) \cap (U_{\nu}^k)'} \hat{\phi}_{\nu}^k dv_{[a'],r} = \sum_{k=1}^{\mathfrak{s}_a} \int_{V_{[a]}^k(r) \setminus T} \phi dv_{[a],r}.$$

Therefore

$$A = \frac{1}{r^{2m}} \int_{U_{[a]}(r)} \phi v_{p[a]}^m = \langle \phi | U \rangle_{a,r}, \quad 0 < r < r_0.$$

Similarly, if $\langle \phi | U \rangle_{a,r} = A$, $\forall r \in (0, r_0)$, then $[\phi]U_{a,r} = A$, $\forall r \in (0, r_0)$. \square

Of importance to harmonic function theory is the *Dirichlet product*, which, on a semi-Riemann domain, can be defined as follows: if $\eta, \phi : D \rightarrow \mathbb{C}$ are locally Lipschitz functions ([14], §4), set

$$(3.6) \quad [\eta, \phi]_D := \int_D d\eta \wedge d^c \bar{\phi} \wedge v_p^{m-1},$$

provided the integral exists. (For further properties and applications of this product, see [15]). The definition (2.3) and the Stokes' theorem ([14], (7.1.3)) imply the following

Lemma 3.2. *Let $\eta, \phi \in C^1(D)$ and $a \in D^0$. Then for every neighborhood U of a and $r_0 > 0$ such that $U_{[a]}(r_0) \subset\subset D$,*

$$(3.7) \quad [\phi]U_{a,r} = \langle \phi | U \rangle_{a,r} + \frac{1}{r^{2m}} [\phi, \|p^{[a]}\|^2]_{a,r}, \quad \forall r \in (0, r_0).$$

A function $\phi \in C^0(D)$ is called (locally) *nearly harmonic at $a \in D^0$* iff there exists a pseudo-ball $U \subseteq D$ at a of radius $r_0 > 0$ such that

$$(3.8) \quad [\phi]U_{a,r} = \langle \phi | U \rangle_{a,r}, \quad \forall r \in (0, r_0).$$

Lemma 3.3. *A function $\phi \in C^0(D)$ is nearly harmonic at point $a \in D^0$ iff there exists a pseudo-ball $U \subseteq D$ at a of radius $r^* > 0$ such that*

$$[\phi]U_{a,r} = \text{const.}, \quad \forall r \in (0, r^*).$$

Proof. Assume ϕ is nearly harmonic at point $a \in D^0$. It suffices to consider the case where ϕ is real-valued. In terms of the Euclidean volume elements the condition (3.8) can be written

$$\sum'_{\nu} \int_{U_{[a]}(r)} \phi_{\nu}^k dv_{[a],r} = \frac{r}{2m} \sum'_{\nu} \int_{dU_{[a]}(r)} \phi_{\nu}^k d\sigma_{[a],r}.$$

Hence by the formula (3.5),

$$\sum'_{\nu} \int_{dU_{[a]}(r)} \phi_{\nu}^k d\sigma_{[a],r} = \sum'_{\nu} \frac{d}{dr} \left(\frac{r}{2m} \int_{dU_{[a]}(r)} \phi_{\nu}^k d\sigma_{[a],r} \right).$$

Thus

$$\frac{d}{dr} \sum'_{\nu} \left(\int_{dU_{[a]}(r)} \phi_{\nu}^k d\sigma_{[a],r} \right) = \frac{(2m-1)}{r} \sum'_{\nu} \int_{dU_{[a]}(r)} \phi_{\nu}^k d\sigma_{[a],r}.$$

Hence one has

$$\frac{d}{dr} \left(\int_{dU_{[a]}(r)} \phi d\sigma_{[a],r} \right) = \frac{(2m-1)}{r} \int_{dU_{[a]}(r)} \phi d\sigma_{[a],r}.$$

It follows that for some $r^* > 0$,

$$\frac{1}{|\mathbb{S}| r^{2m-1}} \left\| \int_{dU_{[a]}(r)} \phi d\sigma_{[a],r} \right\| = \text{const.}, \quad \forall r \in (0, r^*).$$

Therefore the function $\|[\phi]U\|_{a,r}$, hence also $[\phi]U_{a,r}$, is constant for such r . The converse assertion is an immediate consequence of Lemma 3.1. \square

Proposition 1. *If $U \subseteq X$ is a pseudo-ball at $a \in X^0$ and $\phi \in C^0(\overline{U})$ such that $[\phi]U \cap p^{-1}(p(a)) = \text{const.}$, then*

$$(3.9) \quad \deg(p]U) \phi(a) = \lim_{r \rightarrow 0} [\phi]U_{a,r} = \lim_{r \rightarrow 0} \langle \phi]U \rangle_{a,r}.$$

Proof. Without loss of generality assume $U = X$, and let $\tau := \|p^{[a]}\|^2 : X \rightarrow [0, \infty)$. The identity (3.7) implies that $[1]_{a,r} = \langle 1 \rangle_{a,r}$, $\forall r > 0$. Also, it follows from [14], Proposition 5.2.2 and the sheet number formula (2.4) that

$$(3.10) \quad \langle 1 \rangle_{a,r} = \deg(p) \langle 1] \Omega \rangle_{a',r} = \deg(p),$$

where the integral $\langle 1] \Omega \rangle_{a',r} = 1$ is defined in terms of the Euclidean volume element v^m on \mathbb{C}^m . Set $\tilde{\phi} := \phi - \phi(a)$. For each $\varepsilon > 0$, let W_{ε}^{α} be a neighborhood of $z_{\alpha} \in \tau^{-1}(0)$ such that $\|\tilde{\phi}(z)\| < \varepsilon$, $\forall z \in W_{\varepsilon}^{\alpha}$. Take $r_0 > 0$ and choose an open covering of $p^{-1}(a') \cap X_{[a]}[r_0]$ by the open sets W_{ε}^{α} . Then there exists a constant $\delta > 0$ such that $\tau^{-1}([-\delta, \delta]) \cap X_{[a]}[r_0] \subseteq W_{\varepsilon} := \cup_{\alpha} W_{\varepsilon}^{\alpha}$ ([14], (1.1.5)). This implies that $\sup_{X_{[a]}[\delta]} \|\tilde{\phi}\| \leq \varepsilon$. Hence

$$(3.11) \quad \|\langle \tilde{\phi} \rangle_{a,r}\| \leq \sup_{X_{[a]}[\delta]} \|\tilde{\phi}\| \langle 1 \rangle_{a,r} \leq \varepsilon \deg(p), \quad \forall r \in (0, \delta].$$

Therefore it follows from the relations (3.10) and (3.11) that

$$\lim_{r \rightarrow 0} \langle \phi]U \rangle_{a,r} = \phi(a) \deg(p).$$

The remaining assertion on the spherical mean-value is similarly proved. \square

Proposition 2. (*Maximum principle*) *Let D be a domain in X . Assume: i) either $D \subseteq X^0$ or D is irreducible; ii) $\phi : D \rightarrow [-\infty, \infty)$ is upper-semicontinuous and bounded above; iii) $\forall a \in D^0$, there exists a neighborhood $U \subseteq D$ of a such that*

$$(3.12) \quad \nu_p(a) \phi(a) \leq \langle \phi]U \rangle_{a,r} \quad \forall r \in (0, r_0).$$

Then ϕ satisfies the maximum principle on D : if for some $z_0 \in D^0$, $\phi(z_0) = K : = \sup \{\phi(z) \mid z \in D\} \neq -\infty$, then $\phi = \text{constant}$.

Proof. Let $M = \{z \in D \mid \phi(z) < K\}$. For any $a_0 \in D^0 \setminus M$, choose a neighborhood $U \subseteq D^0$ such that the inequality (3.12) holds. Without loss of generality, assume that U is a pseudo-ball at a_0 of radius ρ . Suppose $M \cap U_{[a_0]}(\rho) \neq \emptyset$. Since $\phi(z) < K$ for all z in a neighborhood of each $z^* \in M \cap U_{[a_0]}(\rho)$, it follows from [14], Proposition 5.2.2, and (2.4) that

$$\begin{aligned} \rho^{2m} \nu_p(a_0) \phi(a_0) &\leq \int_{U_{[a_0]}(\rho)} \phi v_p^m \\ &< \int_{U_{[a_0]}(\rho)} K v_p^m = \rho^{2m} \deg(p]U) K. \end{aligned}$$

Hence by the relation (2.5), this implies that $\phi(a_0) < K$, a contradiction. Therefore $M \cap U_{[a_0]}(r) = \emptyset$. Thus the set $D^0 \setminus M$ is open and non-void. It follows from the connectedness of D^0 that $M \cap D^0 = \emptyset$. Consequently $\phi(z) = K$ in D^0 , hence also in D . \square

4. SEMI-HARMONICITY

For later use set $C^\lambda(\overline{D}) := A^{0,\lambda}(\overline{D})$, $C^{1,1}(\overline{D}) := \{\phi \in C^1(\overline{D}) \mid \phi_{\bar{x}_j} \text{ and } \phi_{\bar{y}_j} \in C^\lambda(\overline{D}), 1 \leq j \leq m\}$, and $C^{\lambda,(c)}(\overline{D}) := \{\eta \in C^1(\overline{D}) \mid \eta \rfloor \partial D = \text{const.}\}$. Denote by $C_0^\mu(D)$ the set of compactly supported C^μ -functions. The sets $C^\lambda(\partial D)$, and $C^{1,1}(D)$, $C_0^{1,1}(D)$ are similarly defined. Let $\rho \in A^{2m,0}(D)$. A function f on X is said to be *locally integrable* ($f \in L_{\text{loc}}^1(X)$) provided so is every $2m$ -form $f\chi$ with $\chi \in A^{2m,0}(X)$. Similarly define $L_{\text{loc}}^2(X)$ (with $|g|^2\chi$ in place of $g\chi$). A *weak solution* of the semi-Poisson equation

$$(4.1) \quad dd^c(\phi v_p^{m-1}) = \rho$$

is a locally integrable function $\phi : D \rightarrow \mathbb{C}$ such that $\forall C^2$ -function $u : D \rightarrow [0, \infty)$ with compact support in D^* ,

$$\int_D \phi dd^c u \wedge v_p^{m-1} = \int_D \rho u v_p^m.$$

A (*strong*) *solution* of the equation (4.1) is a C^2 -function $\phi : D \setminus A \rightarrow \mathbb{C}$ for some thin analytic subset A of D such that ϕ satisfies the equation (4.1) (pointwise) in $D \setminus A$. The Stokes theorem shows that every strong solution in $C^{1,1}(\overline{D})$ of the equation (4.1) is a weak solution. It will be shown that on a Riemann domain, if ∂D is reasonably smooth, a bounded weak solution in $C^0(\overline{D})$ of the semi-Poisson equation depends continuously on its boundary values on ∂D .

A locally integrable function $\phi : D \rightarrow \mathbb{C}$ is called: (1) *weakly harmonic in D* (with respect to p) iff ϕ is a weak solution of the semi-Laplace equation

$$(4.2) \quad dd^c(\phi v_p^{m-1}) = 0$$

in D ; (2) *semi-harmonic in D* (with respect to p) iff there exists a thin analytic subset A of D such that ϕ is a C^2 -solution of the semi-Laplace equation (4.2) in $D \setminus A$. Also, $\phi : D \rightarrow \mathbb{C}$ is called *semi-harmonic at $a \in D$* iff ϕ is semi-harmonic in a neighborhood of a . The real and imaginary parts of every weakly holomorphic or pluri-harmonic function in D are semi-harmonic. Locally every pure m -dimensional complex space X is an analytic covering of a domain in \mathbb{C}^m . Thus by means of the identity (2.10), the *semi-harmonicity* of a function ϕ on a domain $D \subseteq X$ is intrinsically defined by the requirement (4.2) in terms of the local covering maps of X . Let $\mathfrak{H}(D)$, resp. $\mathfrak{H}_w(D)$, denote the set of all semi-harmonic, resp. weakly harmonic, functions in D .

Theorem 4.1. *Let (X, p) be a Riemann domain and $\phi \in C^0(X)$. The following conditions are equivalent:*

- (1) ϕ is locally nearly harmonic in X .
- (2) ϕ has the local spherical mean-value property in X .
- (3) ϕ has the local solid mean-value property in X .

Proof. By Lemma 3.3, the near-harmonicity at $a \in X$ of ϕ is equivalent to the constancy of its local spherical mean-value at a . Hence it follows then from Lemma 3.1, Proposition 1 and the relation (2.5) that at each point of X , the above three conditions on ϕ are equivalent. \square

An element $\phi \in L^1_{\text{loc}}(X)$ is said to be *locally forwardly square-integrable* ($\phi \in L^2_{\text{loc}}[X]$) if there exists a pseudoball $U \Subset X$ at each point of X such that the following integral exists: $\int_{U' \setminus \Delta'} \|(\sum'_\nu \hat{\phi}_\nu^j)(z')\|^2 dv(z') < \infty$, where the (measurable) function $\sum'_\nu \hat{\phi}_\nu^j : U' \setminus \Delta' \rightarrow \mathbb{C}$ is defined by: $z' \mapsto \lim_{n \rightarrow \infty} \sum_{l=1}^{N_n^j} \hat{\phi}_{n_l}^j(z')$, the summation being taken over a covering of $K_n \cap V^j$ by the open sets $U_{n_l}^j$, $l = 1, \dots, N_n^j$, the $\{K_n\}$ being an exhausting sequence of compact subsets of U^* . Note that it can be shown (by using [14, Theorem 5.2.2]) that the function $\sum'_\nu \hat{\phi}_\nu^j$ is integrable on $U' \setminus \Delta'$. The following characterization of semi-harmonicity also gives a criterion for the removability (in a weak sense) of analytic singularities:

Theorem 4.2. *Let (X, p) be a Riemann domain. The following assertions "(2) \Rightarrow (3) \Rightarrow (1)" hold; moreover, the implication "(1) \Rightarrow (2)" is valid if $\phi \in L^2_{\text{loc}}[X]$:*

- (1) $\phi \in \mathfrak{H}_w(X)$.
- (2) ϕ is locally integrable in X and defines a current $[\phi]$ induced by a function $\tilde{\phi} \in C^{\beta \cap m}(X) \cap C^0(X^*)$ satisfying the local solid mean-value property in the domain of continuity of $\tilde{\phi}$.
- (3) $\phi \in \mathfrak{H}(X)$.

Proof. To prove the assertion "(1) \Rightarrow (2)", let $U \subseteq X$ be a pseudo-ball of radius r_0 at a point $a \in X$. Choose a non-negative function $\alpha \in C_0^\infty(\mathbb{R}^{2m})$ with support contained in $\mathbb{B}[1]$ such that $\int_{\mathbb{R}^{2m}} \alpha dv = 1$. Set $\alpha_\varepsilon(x) := \varepsilon^{-2m} \alpha(\frac{x}{\varepsilon})$, $\alpha_\varepsilon^{(0)}(x) = \alpha_\varepsilon(-x)$, $\forall \varepsilon > 0$. Define $\hat{\phi}_{j,\varepsilon} = \alpha_\varepsilon * (\phi|V^j)$ for each branch V^j , $1 \leq j \leq \mathfrak{s}_a$, of U , by setting

$$\hat{\phi}_{j,\varepsilon}(x) := \int_{V^j} \alpha_\varepsilon(x - y') \phi(y) d\tilde{v}$$

on $W_\varepsilon = \{x \in U' \mid \text{dist}(x, \partial U') > \varepsilon\}$, $d\tilde{v} := p^*(dv)$ being the pullback of the Euclidean volume element on \mathbb{C}^m . Then $\hat{\phi}_{j,\varepsilon} \in C^\infty(W_\varepsilon)$. Let $U'' \subset \subset W \subset \subset p(V^j)$

be open subsets of U' . Set $\varepsilon_1 = \frac{1}{2} \text{dist}(\overline{U''}, \partial W)$. Then $\forall w \in C_0^\infty(Q)$, $Q = U_\nu^j$ being a local sheet of $p^{-1}(U'') \cap V^j$ (at a point z), and denoting by \hat{w} (respectively, $\hat{\phi}^{j,\nu}$) the function on Q' induced by w (respectively, $\phi|_Q$), one has $\text{Spt}(\alpha_\varepsilon * \hat{w}) \subseteq W$, $\forall \varepsilon \in (0, \varepsilon_1)$ and

$$(4.3) \quad \begin{aligned} \int_{Q'} \hat{\phi}_{j,\varepsilon} dd^c \hat{w} \wedge v_{[z']}^{m-1} &= \text{const.} (\hat{\phi}^{j,\nu}, \alpha_\varepsilon^{(0)} * \Delta \hat{w})_{Q'} \\ &= \text{const.} (\hat{\phi}^{j,\nu}, \Delta(\alpha_\varepsilon^{(0)} * \hat{w}))_{Q'}. \end{aligned}$$

Therefore

$$\int_{Q'} \hat{\phi}_{j,\varepsilon} dd^c \hat{w} \wedge v_{[z']}^{m-1} = \int_{U''} \hat{\phi}^{j,\nu} dd^c (\alpha_\varepsilon^{(0)} * \hat{w}) v_{[z']}^{m-1}.$$

Thus the weak harmonicity of $\hat{\phi}^{j,\nu}$ implies that each $\hat{\phi}_{j,\varepsilon}$ is weakly harmonic in V^j for sufficiently small $\varepsilon > 0$. Hence for any $\hat{u} \in C_0^2(U'')$ with $\hat{u} \geq 0$ and $\hat{u} = 1$ in a neighborhood of a point $z' \in U'' \setminus \Delta'$, the relation

$$(4.4) \quad \int_{U''} \hat{\phi}_{j,\varepsilon} dd^c \hat{u} \wedge v_{[z']}^{m-1} = \int_{U''} \hat{u} dd^c \hat{\phi}_{j,\varepsilon} \wedge v_{[z']}^{m-1} = 0$$

(for sufficiently small $\varepsilon > 0$) implies that the function $\hat{\phi}_{j,\varepsilon}$ is harmonic in U'' . Observe that the Dirichlet product

$$[\hat{\phi}_{j,\varepsilon}, \|z' - a'\|^2]_{a',r} = \int_{U'_{a'}(r)} (r^2 - \|z' - a'\|^2) dd^c \hat{\phi}_{j,\varepsilon} \wedge v_{[a']}^{m-1}$$

for small $r > 0$. Hence the identity (3.7) implies that $\hat{\phi}_{j,\varepsilon}$ is nearly harmonic at a' . Thus it follows from Lemma 3.3 and the relation (3.9) that there exists $r_{a'} \in (0, r_0)$ such that

$$\hat{\phi}_{j,\varepsilon}(a') = \langle \hat{\phi}_{j,\varepsilon} \rfloor \mathbb{B}_{[a']}(r) \rangle_{a',r}, \quad \forall r \in (0, r_{a'}).$$

Here the $r_{a'}$ may be chosen to be independent of ε (for small ε), since the open set W_ε increases with decreasing ε . Assume at first that $\phi \in C^0(X)$. Letting $\varepsilon \rightarrow 0$ this relation yields

$$\lim_{\varepsilon \rightarrow 0} \langle \hat{\phi}_{j,\varepsilon} \rfloor \mathbb{B}_{[a']}(r) \rangle_{a',r} = \lim_{\varepsilon \rightarrow 0} \hat{\phi}_{j,\varepsilon}(a') = s_j \phi(a), \quad \forall r \in (0, r_{a'}),$$

the s_j being the sheet number of $p|V^j$ (where the last equality follows from [14, Theorem 5.2.2]). Suppose that $f \in C_0^\infty(U_\nu^j)$ and $r \in (0, r_{a'})$. There exists an $\varepsilon_r > 0$ such that $U'_{[y]}(\varepsilon) \subseteq U'_{[a]}(r)$ for all $y \in V_{[a]}^j(r) \cap \text{Spt}(f)$. Then for all $\varepsilon \in (0, \varepsilon_r)$,

$$\begin{aligned}
\langle f \rfloor \mathbf{1}_{V^j} \rangle_{a,r} &= \frac{1}{r^{2m}} \int_{V_{[a]}^j(r) \cap U_{\nu}^j} f(y) \cdot \left(\int_{z' \in U' \setminus \Delta'} \alpha_{\varepsilon}^{(0)}(y' - z') dv(z') \right) v_p^m(y) \\
&= \frac{1}{r^{2m}} \int_{(U_{\nu}^j)_{[a]}(r)} \left(\int_{V_{[a]}^j(r) \cap U_{\nu}^j} (\alpha_{\varepsilon}^0 \circ p^{[z]})(y) f(y) d\tilde{v}(y) \right) v_p^m(z) \\
&= \langle f_{\{j,\varepsilon\}} \rfloor \mathbf{1}_{\mathbb{B}_{[a']}(r)} \rangle_{a',r}.
\end{aligned}$$

Denoting by $\psi_{\nu,\varepsilon}^j$ the convolution $\alpha_{\varepsilon} * (\rho_{\nu}^j \phi)$, one has $\hat{\phi}_{j,\varepsilon} = \sum_{\nu}' \psi_{\nu,\varepsilon}^j$, so that

$$\langle \phi \rfloor \mathbf{1}_{V^j} \rangle_{a,r} = \sum_{\nu} \langle \psi_{\nu,\varepsilon}^j \rfloor \mathbb{B}_{[a']}(r) \rangle_{a',r} = \langle \hat{\phi}_{j,\varepsilon} \rfloor \mathbb{B}_{[a']}(r) \rangle_{a',r}.$$

Thus

$$\langle \phi \rfloor U \rangle_{a,r} = \sum_j \langle \phi \rfloor \mathbf{1}_{V^j} \rangle_{a,r} = \sum_j s_j \phi(a) = \nu_p(a) \phi(a), \quad \forall r \in (0, r_{a'}).$$

Hence the local solid mean-value property holds for ϕ .

The general case will be proved by adapting the proof of Gårding [10]. Let $\gamma(\xi, z)$, $\xi, z \in U$ with $\xi' \neq z'$, be the Newtonian potential

$$(4.5) \quad \gamma(\xi, z) := \begin{cases} \frac{1}{2\pi} \log \|\xi' - z'\|^2, & \text{if } m = 1, \\ \frac{-1}{(2m-2)|\mathbb{S}| \|\xi' - z'\|^{2m-2}}, & \text{if } m > 1. \end{cases}$$

Suppose that $\phi \in L_{\text{loc}}^2[X]$ and let $p : U = U_{[a]}(r_0) \rightarrow U' = \mathbb{B}(r_0)$ be a pseudoball exhibiting the local forward square-integrability of ϕ . Let G'_1, G'_2 be open sets with $G'_1 \subset \subset G'_2$ where $\overline{G'_2} \subset U'_1 := \mathbb{B}(r_1)$, $r_1 < r_0$. Choose $\beta' \in C^\infty(U')$ such that $\beta = 0$ in a neighborhood of $\overline{G'_1}$ and $\beta' \rfloor U \setminus \overline{G'_2} = 1$. Let $G_j := p^{-1}(G'_j)$, $j = 1, 2$, and $\beta := p^* \beta'$ on U . Then the function

$$h(\xi) := \frac{4\pi^m c_m}{(m-1)!} \int_{\overline{G_2} \setminus G_1} \phi dd^c(\beta(z)\gamma(\xi, z)) \wedge v_p^{m-1}, \quad \forall \xi \in G_1,$$

is well-defined. Choose a C^∞ -partition of unity $\{(U_{\nu}^j, \rho_{\nu}^j)\}$ on $V^j \cap \hat{U}$ in terms of the open sets $U_{\nu}^j \subseteq V^j$. Let γ' be the function induced by γ on $U' \times U'$. For fixed $\xi \in G_1$ the form $\Delta_{z'}(\beta'(z')\gamma'(\xi', z')) dv(z')$ is compactly supported in a neighborhood of $\overline{G_2}$. The integrability of $\sum_{\nu}' \hat{\phi}_{\nu}^j$ on $U' \setminus \Delta'$ implies that of the function

$$f_j(\xi', z') = \left(\sum_{\nu}' \hat{\phi}_{\nu}^j \right)(z') \Delta_{z'}(\beta'(z')\gamma'(\xi', z')) \hat{\rho}(\xi')$$

is integrable on the product space $G'_1 \times U'_{[a']}[r_1]$ for any $\hat{\rho} \in C^\infty(U'(r_1))$ (for $r_1 < r_0$). Thus the function h , being equal to the sum of integrals (with respect to z') over U' of functions of the above type (with $\hat{\rho} = 1, j = 1, \dots, s_a$), belongs to $C^m(G_1) \cap L_{\text{loc}}^1(G_1^*)$, by the Fubini's Theorem. Note that by virtue of the L^2 -integrability of $\sum_{\nu}' \hat{\phi}_{\nu}^j$ and the Hölder's inequality, one has, for $\xi \in G_1$,

$$\|h(\xi)\| \leq \sum_{j=1}^{s_a} \left[\int_{U'} \left\| \sum_{\nu}' \hat{\phi}_{\nu}^j(z') \right\|^2 dv(z') \right]^{\frac{1}{2}} \left[\int_{U'} \|\Delta_{z'}(\beta'(z')\gamma'(\xi', z'))\|^2 dv(z') \right]^{\frac{1}{2}}$$

thus proving that h is bounded on G_1 . Hence it follows from [14, Lemma 4.2.12] that h is locally integrable in G_1 . Since h is C^∞ and harmonic in G_1^* , it is semi-harmonic in G_1 . It will be shown that $\forall \eta \in C_0^\infty(G_1^*)$,

$$(h, \eta)_{G_1} = (\phi, \eta)_{G_1}.$$

It suffices to prove the following:

$$(4.6) \quad \int_{G_1} h \eta v_p^m = \int_{G_1} \phi \eta v_p^m,$$

where $\eta \in C_0^\infty(U_{\nu_0}^{j_0} \cap G_1^*)$. By an interchange of the order of integration on the left-hand side of (4.6), this equation can be expressed equivalently as follows:

$$(4.7) \quad \int_{G_1} h(\xi) \eta(\xi) d\tilde{v} = \sum_{j=1}^{s_a} \int_{G_2' \setminus \Delta'} \left(\sum_{\nu}' \hat{\phi}_{\nu}^j(z') \right) \Delta_{z'} \psi(z') dv(z'),$$

where

$$\psi(z') := \beta'(z') \int_{(U_{\nu_0}^{j_0})' \cap G_1'} \gamma_{\nu_0}^{j_0}(\xi', z') \hat{\eta}(\xi') dv(\xi'), \quad z' \in \overline{G_2'},$$

(where γ_{ν}^j is induced by γ on $(U_{\nu}^j)' \times (U_{\nu}^j)'$, and the $\hat{\eta}$ by η on $(U_{\nu_0}^{j_0})'$, the operation being justified by the Fubini's Theorem (for the same reason as given above). Observe that

$$\Delta_{z'} \psi(z') = -\Delta_{z'}(u(z')) + \Delta_{z'} \left(\int_{(U_{\nu_0}^{j_0})' \cap G_1'} \gamma_{\nu_0}^{j_0}(\xi', z') \hat{\eta}(\xi') dv(\xi') \right),$$

where the function

$$u(z') := (1 - \beta'(z')) \int_{(U_{\nu_0}^{j_0})' \cap G_1'} \gamma_{\nu_0}^{j_0}(\xi', z') \hat{\eta}(\xi') dv(\xi')$$

belongs to $C_0^\infty(U_1')$. Thus

$$(4.8) \quad \Delta_{z'} \psi(z') = \begin{cases} \hat{\eta}(z') - \Delta_{z'}(u(z')), & \text{if } z' \in G_1' \cap (U_{\nu_0}^{j_0})', \\ 0 - \Delta_{z'}(u(z')), & \text{if } z' \in G_1' \setminus (U_{\nu_0}^{j_0})'. \end{cases}$$

Hence the assertion (4.6) follows from the relations (4.7)-(4.8) and the weak harmonicity of ϕ . Consequently $h = \phi$ almost everywhere in G_1^* . Since G_1' is an arbitrary relatively compact open subset of U' , there exists a semi-harmonic function $\tilde{\phi} \in C^{\beta \cap m}(X) \cap C^\infty(X^*)$ such that $\tilde{\phi} = \phi$ a. e. in X . Hence, by what has been proved, $\tilde{\phi}$ satisfies the local solid mean-value property in the domain of continuity of $\tilde{\phi}$.

To prove the assertion "(2) \Rightarrow (3)", let U be a pseudo-ball at a point $a \in X^*$ of radius r_0 and $0 < r < r_0$. Without loss of generality, assume that $\tilde{\phi}$ is real-valued. There exists a continuous function $\hat{\phi}$ on the closed ball $\mathbb{B}_{[a]}[r]$ such that

$p^*\hat{\phi} = \tilde{\phi}$ on $U \cap p^{-1}(\mathbb{B}_{[a']}[r])$. Also, there exists a continuous function h on $\mathbb{B}_{[a']}[r]$, harmonic in $\mathbb{B}_{[a']}(r)$, such that $h(w) = \hat{\phi}(w)$, $\forall w \in \mathbb{S}_{[a']}(r)$. Then the function $\psi := \hat{\phi} - h$ has the solid mean-value property at a' and vanishes on $\mathbb{S}_{[a']}(r)$. The same is true for the function $-\psi$. Hence it follows from the maximum principle (Proposition 2) that $\psi \equiv 0$ on $\mathbb{B}_{[a']}(r)$. Consequently ϕ is semi-harmonic in X .

To prove the assertion "(3) \Rightarrow (1)", let U, U_0 be pseudo-balls at $a \in X^*$ with $U_0 \subset\subset U \subset\subset X^*$. It follows as in (4.3) that $\forall w \in C_0^0(U'_0)$,

$$\int_{U'_0} \hat{\phi}_\varepsilon \Delta w dv = \text{const.} \int_W dd^c \hat{\phi} \wedge (\alpha_\varepsilon^{(0)} * w) v_{[a']}^{m-1},$$

for sufficiently small $\varepsilon > 0$, where the $\hat{\phi}$ being induced by ϕ on U'_0 . By the semi-harmonicity of ϕ , the above last integral vanishes. This shows that $\hat{\phi}_\varepsilon$ is harmonic in U' for sufficiently small $\varepsilon > 0$. Let $u : X \rightarrow [0, \infty)$ be a C^2 -function with compact support in X^* . By using a partition of unity, it may be assumed that $\text{Spt}(u)$ is contained in a pseudo-ball $U_0 \subset\subset X^*$. Then the following relation holds (for $\hat{\phi}_\varepsilon$ and the induced $\hat{u} \in C_c^2(U')$ of u):

$$(4.9) \quad (\hat{u}, \Delta \hat{\phi}_\varepsilon)_{U''} = (\hat{\phi}_\varepsilon, \Delta \hat{u})_{U''} = 0$$

It follows then from this relation (and the expression for the semi-Laplace operator) that

$$\left| \int_{U'_0} \hat{\phi} dd^c \hat{u} \wedge v_{[a']}^{m-1} \right| \leq \text{Const.} \int_{U'_0} |\hat{\phi} - \hat{\phi}_\varepsilon| v_{[a']}^m.$$

The L^1 -convergence of $\hat{\phi}_\varepsilon$ to $\hat{\phi}$ implies that

$$\int_{U'_0} \hat{\phi} dd^c \hat{u} \wedge v_{[a']}^{m-1} = 0.$$

Therefore ϕ is weakly harmonic in X . \square

Remark 1. If $p : D \rightarrow \Omega$ is a Riemann domain, an element $\phi \in \mathfrak{H}_w(D) \cap L_{\text{loc}}^2[D]$ will be identified with its representative $\tilde{\phi} \in C^{\beta \cap \mathfrak{m}}(D) \cap C^\infty(D^*)$. If ϕ is locally integrable in D and $\phi \in C^0(D^*)$, then ϕ is semi-harmonic in D iff in D^* the local near harmonicity or the solid, resp. spherical, mean-value property holds for ϕ .

The above Theorem gives an extension of the Weyl's Lemma ([17], pp. 415-416) to a Riemann domain:

Corollary 4.1. *If (X, p) is a Riemann domain, then*

$$\mathfrak{H}_w(X) \cap L_{\text{loc}}^2[X] = C^\beta(X) \cap C^\infty(X^*) \cap \ker(\Delta_p]X^*).$$

Theorem 4.2 and the maximum principle imply the following

Corollary 4.2. *Assume (X, p) is a Riemann domain. Let D be a domain in X with $dD \neq \emptyset$. Assume: i) either $D \subseteq X^0$ or D is irreducible; (ii) $\eta_j \in C^0(\partial D)$ for $j = 1, 2$ and $|\eta_1 - \eta_2| < \varepsilon$ on ∂D ; (iii) $\phi = \phi_j \in C^0(\overline{D})$, $j = 1, 2$, are bounded, weak solutions to the Dirichlet problem*

$$dd^c(\phi v_p^{m-1}) = \rho \text{ in } D \setminus A, \quad \phi \rfloor dD = \eta_j,$$

where A is thin analytic in D , and $\rho \in A^{2m,0}(D \setminus A)$. Then $|\phi_1 - \phi_2| < \varepsilon$ on \overline{D} .

5. EULER AND NEUMANN VECTOR FIELDS

For each $a \in X$, define $\rho = \rho_a := \|p^{[a]}\| : X \rightarrow \mathbb{R}$. The associated ∂ -, respectively, $\bar{\partial}$ -, Euler vector field (multiplied by ρ_a) are given by

$$(5.1) \quad E_{p,a} := \rho_a \mathcal{E}_{\rho_a}; \quad \bar{E}_{p,a} := \rho_a \bar{\mathcal{E}}_{\rho_a}$$

in X^* . It is easily shown that

$$(5.2) \quad \bar{E}_{p,a} = \sum_{j=1}^m (\bar{p}_j - \bar{p}_j(a)) \frac{\partial}{\partial \bar{p}_j} \text{ in } X^*.$$

If $\phi \in C^1(D)$ and $a \in D$, the a -radial derivative of ϕ is given by

$$(5.3) \quad (\mathfrak{R}_{p,a}\phi)(z) := \partial_{\nabla_{\rho_a}}(\phi)(z), \quad z \in D^*.$$

Then $\forall z \in D^* \setminus p^{-1}(a')$,

$$(5.4) \quad (\mathfrak{R}_{p,a}\phi)(z) = \sum_{j=1}^m \frac{(\tilde{x}_j - \tilde{x}_j(a)) \frac{\partial \phi}{\partial \tilde{x}_j} + (\tilde{y}_j - \tilde{y}_j(a)) \frac{\partial \phi}{\partial \tilde{y}_j}}{\|p^{[a]}(z)\|} \Big|_z$$

(cf. [6], p. 169).

Let $j_{aD} : dD \rightarrow X$, and $j_{a,r} : dD_a(r) \rightarrow X$, denote the inclusion mapping, for $a \in D^*$ and small positive r . By tedious calculations it can be shown that

$$(5.5) \quad j_{a,r}^* \left(\left(\frac{i}{2\pi} \right) \bar{\partial} \phi \wedge v_p^{m-1} \right) (z) = r^{2m-2} \bar{E}_{p,a}(\phi) \Big|_z \sigma_a, \quad \forall z \in dD_a(r).$$

It follows from the identities (5.3)-(5.5) that, $\forall z \in D^* \setminus p^{-1}(a')$,

$$(5.6) \quad j_{a,r}^* (d^c \phi \wedge v_p^{m-1}) (z) = \frac{r^{2m-1}}{2} (\mathfrak{R}_{p,a}\phi)(z) \sigma_a.$$

The following Proposition shows that the *normal derivative* of $\phi \in C^1(\overline{D})$ on dD can be intrinsically defined: Let $d\sigma_{aD}$ denote the (Lebesgue) surface measure on dD induced by the local patches $p_U := p : U \rightarrow \mathbb{B}_{[a']}(r_0)$, at a point $a \in X^* \cap dD$, and orientation-preserving diffeomorphisms of $\mathbb{B}_{[a']}(r_0)$.

Proposition 3. *Let $\rho = 0$ be a local C^1 defining equation of dD in a neighborhood $U \subseteq X^*$ of $a \in dD \cap X^*$ with $d\rho \neq 0$ on $dD \cap U$. Then $\forall \phi \in C^1(\overline{D})$,*

$$(5.7) \quad j_{aD}^* (d^c \phi \wedge v_p^{m-1}) = (-1)^{\frac{m(m-1)}{2}} \frac{1}{2\|\mathbb{S}\|} \partial_\nu \phi d\sigma_{aD},$$

where $\nu := \frac{\nabla \rho}{\|\nabla \rho\|}$, $\|\nabla \rho\|$ being the induced Euclidean norm of $\nabla \rho$.

Proof. Let $dp_{[j]} := dp_1 \wedge \cdots \wedge dp_{j-1} \wedge dp_{j+1} \wedge \cdots \wedge dp_m$ and $d\bar{p} := d\bar{p}_1 \wedge \cdots \wedge d\bar{p}_m$. It can be shown (by tedious computations) that

$$(5.8) \quad \begin{aligned} j_{U \cap dD}^* (dp_{[j]} \wedge d\bar{p}) &= 2^m i^m (-1)^{m+j-1} \rho_j d\sigma_{U \cap dD}, \\ j_{U \cap dD}^* (dp \wedge d\bar{p}_{[j]}) &= 2^m i^m (-1)^{j-1} \rho_{\bar{j}} d\sigma_{U \cap dD}, \end{aligned}$$

where $\rho_j := (\partial\rho/\partial p_j) \|\nabla\rho\|^{-1}$ and $\rho_{\bar{j}} := (\partial\rho/\partial \bar{p}_j) \|\nabla\rho\|^{-1}$. It follows from the definition (2.1) that

$$d^c \phi \wedge v_p^{m-1} = (-1)^{\frac{(m-1)(m-2)}{2}} \frac{\left(\frac{i}{2}\right)^m}{\|\mathbb{S}\|} \sum_{j=1}^m \phi_{\bar{p}_j} (-1)^{m+j} dp_{[j]} \wedge d\bar{p} - (-1)^{j-1} \phi_{p_j} dp \wedge d\bar{p}_{[j]}.$$

From this the desired conclusion can be deduced by making use of the identities (2.7)-(2.8) and (5.8). \square

A relatively compact open set $G \subseteq X$ is called a *weak Stokes domain* in X iff there exists a thin analytic set A in X containing the singular points of X such that $\partial G \setminus A$ has locally finite \mathfrak{H}^{2m-1} -measure and there exists an (oriented) boundary manifold S of $G \cap \mathcal{R}(X)$ contained in dG such that $\partial G \setminus (A \cup S)$ has zero \mathfrak{H}^{2m-1} -measure. Since the set $dG \setminus (A \cup S)$ is a set of zero \mathfrak{H}^{2m-1} -measure, integration over S and dG make no difference, provided one of them exists (and the notation S will not be explicitly used).

Owing to the normal derivative formula (5.7) and the equations (2.9)-(2.10), the Stokes' theorem ([14], (7.1.3)) yields the following generalized *Green's first identity*:

Lemma 5.1. *If $G \subseteq X$ is a weak Stokes domain, then for all $\eta \in C^\lambda(\overline{G})$ and $\phi \in C^{1,1}(\overline{G})$,*

$$(5.9) \quad [\eta, \bar{\phi}]_G = \int_{dG} \eta d^c \phi \wedge v_p^{m-1} - \int_G \eta dd^c \phi \wedge v_p^{m-1}.$$

Proposition 4. *If $G \subseteq X$ is a weak Stokes domain, then, with respect to the Dirichlet product,*

- (1) $\mathfrak{H}(G) \cap C^{1,1}(G) = C^{1,1}(G) \cap (C_0^\lambda(G; \mathbb{R}))^\perp$;
- (2) $\mathfrak{H}(G) \cap C^{1,1}(\overline{G}) = C^{1,1}(\overline{G}) \cap (C^{\lambda,(c)}(\overline{G}))^\perp$.

Proof. (1) Assume $\phi \in C^{1,1}(G) \cap (C_0^\lambda(G; \mathbb{R}))^\perp$. For each $a \in G^*$ choose a pseudo-ball $U \subset\subset G$ at a of radius r_0 . The function $\eta_{a,r} : \overline{G} \rightarrow \mathbb{R}$, $r \in (0, r_0)$, defined by

$$\eta_{a,r} := \begin{cases} \|p^{[a]}\|^2 - r^2, & \text{if } z \in U_{[a]}(r), \\ 0, & \text{if } z \in \overline{G} \setminus \overline{U_{[a]}(r)}, \end{cases}$$

is locally Lipschitz in G . Thus $[\phi, \|p^{[a]}\|^2]_{a,r} = [\phi, \eta_{a,r}]_G = 0$. By the identity (3.7), the condition that $[\phi, \|p^{[a]}\|^2]_{a,r} = 0$ locally in G^* (i.e. for small $r > 0$) is equivalent to ϕ being locally nearly harmonic in G^* , hence, by Theorems 4.1 and 4.2, also to ϕ being semi-harmonic in G . The converse assertion follows from the Green's identity (5.9)

(2) Let $\phi \in \mathfrak{H}(G) \cap C^{1,1}(\overline{G})$. If $\xi \in C^\lambda(\overline{G})$ and $\xi|_{\partial G} = \text{const.}$, the Green's identity (5.9) implies that $[\xi, \bar{\phi}]_G = 0$. Hence by the hermitian symmetry of the

Dirichlet product ([15], (3.6)), $[\phi, \xi]_G = 0$. Conversely, if $w \in C^{1,1}(\overline{G})$ such that $[w, \xi]_G = 0$, $\forall \xi \in C^\lambda(\overline{G})$ with $\xi \rfloor \partial G = 0$, then by the assertion (1), w is semi-harmonic in G . \square

Proposition 5. *Assume that $G \subset X$ is a weak Stokes domain with $dG \neq \emptyset$, $\rho \in A^{2m,0}(G \setminus A; \mathbb{R})$, where A is thin analytic in G , and $\eta \in C^\lambda(\partial G; \mathbb{R})$. If the Neumann problem*

$$(5.10) \quad dd^c(\phi v_p^{m-1}) = \rho \quad \text{in } G \setminus A, \quad \partial_\nu \phi = \eta \quad \text{on } dG \setminus A$$

admits a real, weak (resp. strong) solution $\phi = \phi_0 \in C^1(\overline{G})$ (resp. $C^{1,1}(\overline{G})$), then the set of all real, weak (resp. strong) solutions in $C^1(\overline{G})$ (resp. $C^{1,1}(\overline{G})$) of the equation (4.1) on $G \setminus A$ subject to the boundary condition

$$(5.11) \quad \partial_\nu \phi \geq \eta \quad (\text{or } \partial_\nu \phi \leq \eta) \quad \text{on } dG \setminus A$$

is given by $\{\phi_0 + \text{const.}\}$

Proof. Since every strong solution in $C^{1,1}(\overline{G})$ to the equation (4.1) is a weak solution, it suffices to consider the case where $\phi = \phi_1 \in C^1(\overline{G})$ is a real, weak solution to the equation (4.1) satisfying the boundary condition (5.11). Set $\chi := \phi_1 - \phi_0$. Since χ is semi-harmonic in G and, by the identity (5.7), $d^c \chi \wedge v_p^{m-1} \rfloor dG \setminus A \geq 0$ (or ≤ 0), one has

$$\int_{dG} d^c \chi \wedge v_p^{m-1} = \int_G dd^c \chi \wedge v_p^{m-1} = 0.$$

It follows that $d^c \chi \wedge v_p^{m-1} \rfloor dG \setminus A \equiv 0$. Then the Green's identity (5.9) yields

$$[\chi, \chi]_G = \int_{dG} \chi d^c \chi \wedge v_p^{m-1} - \int_G \chi dd^c \chi \wedge v_p^{m-1} = 0.$$

Hence $\|\nabla \chi\|^2 = 0$ a.e. in G . Therefore the function χ is locally constant in G^0 . Consequently $\chi = \text{constant}$ on \overline{G} . This completes the proof of the Proposition. \square

Remark 2. The above Proposition implies that, given $\rho \in A^{2m,0}(G \setminus A; \mathbb{C})$ and $\eta \in C^\lambda(\partial G; \mathbb{C})$, an entirely similar assertion holds for the complex Neumann problem (5.10).

Remark 3. Let $G \subseteq X$ be a weak Stokes domain with $dG \neq \emptyset$. If $\phi \in C^1(\overline{G})$ is weakly harmonic in G , then $\phi = \text{const.}$ in \overline{G} iff $\partial_\nu \phi \rfloor dG = \text{const.}$

Example 1. Let P be a real homogeneous polynomial of degree l in the variables $x_1, y_1, \dots, x_{2m}, y_{2m}$. It is well-known that P can be written

$$P = \sum_{j=0}^l \|x\|^{l-j} H_j(x),$$

where H_j is a harmonic homogeneous polynomial of degree j , with $H_j \equiv 0$ whenever $l - j$ is odd. The H_j can be calculated by an effective algorithm (see [3]). Let $\pi : X \rightarrow \mathbb{C}^m$ be an analytic covering with sheet number s . Set $\tilde{P} = \pi^* P$ and $G := \{z \in X \mid \|\pi(z)\| < 1\}$. It follows from the identity (5.7) and Proposition 5 that the set of all the strong solutions in $C^{1,1}(\overline{G})$ of the Neumann problem:

$$dd^c(\phi v_\pi^{m-1}) = 0 \quad \text{in } G^*, \quad \partial_\nu \phi \geq \tilde{P} - H_0 \quad \text{on } dG \cap G^*,$$

is given by $\{\psi + \text{const.}\}$, where $\psi := \sum_{j=1}^l \pi^* H_j / j$. Consequently,

$$\int_{dG} \tilde{P} d\sigma_{dG} = \begin{cases} s H_0 |\mathbb{B}|, & \text{if } l = \text{even}, \\ 0, & \text{if } l = \text{odd}. \end{cases}$$

In [16], p. 182, Weyl gave an alternative definition of the Laplace operator in terms of the Gauss' divergence theorem. In this light it makes sense to define, in view of the identity (5.7), the *harmonic residue* of a function $\phi \in C^1(D \setminus \{a\})$ at a point $a \in D^0$ as follows:

$$(5.12) \quad \text{Res}_a(\phi, r) := \begin{cases} \int_{dD_{[a]}(r)} (-d^c) \phi, & m = 1, \\ \frac{1}{m-1} \int_{dD_{[a]}(r)} (-d^c) \phi \wedge v_p^{m-1}, & m > 1, \end{cases}$$

for small $r > 0$ (cf. Bôcher [5] (see also [7]), and [2], pp. 213-214). If ϕ is a semi-harmonic function with an isolated singularity at a , then the definition (2) is independent of the pseudo-radius (as the Stokes' theorem easily shows).

Example 2. Let $p : X \rightarrow \Omega$ be a Riemann domain and $h \in \mathfrak{H}(X)$. Let $a \in X$ and $\alpha, s \in [0, \infty)$ be constants. Define $\phi : X \setminus p^{-1}(p(a)) \rightarrow \mathbb{C}$ by

$$\phi(z) := \frac{(\log \|p^{[a]}\|^2)^\alpha h(z)}{\|p^{[a]}(z)\|^{2m-2+s}}.$$

Let U be a pseudo-ball at a of radius $r_0 > 0$. Then $\forall r \in (0, r_0)$,

$$(5.13) \quad \text{Res}_a(\phi, r) = \left[-\frac{\alpha (\log r^2)^{\alpha-1}}{r^s} + \frac{\frac{s}{2}}{r^s} (\log r^2)^\alpha \right] \nu_p(a) h(a), \quad m = 1,$$

$$(5.14) \quad \text{Res}_a(\phi, r) = \left[-\frac{\alpha (\log r^2)^{\alpha-1}}{(m-1)r^s} + \frac{m-1+\frac{s}{2}}{(m-1)r^s} (\log r^2)^\alpha \right] \nu_p(a) h(a), \quad m > 1.$$

Proof. Let $k = \frac{s}{2}$, and $r \in (0, r_0)$. Then $\forall \alpha > 0$ one has

$$\begin{aligned} j_{a,r}^*(d^c \phi) &= \frac{(\log r^2)^\alpha j_{a,r}^* d^c h}{r^{2m-2+2k}} + \frac{\alpha (\log r^2)^{\alpha-1} j_{a,r}^*(h d^c \|p^{[a]}\|^2)}{r^{2m+2k}} \\ &\quad - (m-1+k) \frac{(\log r^2)^\alpha j_{a,r}^*(h d^c (\|p^{[a]}(z)\|^2))}{r^{2m+2k}}. \end{aligned}$$

It follows from this relation and the semi-harmonicity of h that, upon integrating the form $d^c \phi \wedge v_p^{m-1}$ over $dU_{[a]}(r)$, the first term vanishes, the second and the third term yield the number $\frac{\alpha (\log r^2)^{\alpha-1}}{r^s} \nu_p(a) h(a)$, respectively, $\frac{-(m-1+k) (\log r^2)^\alpha}{r^s} \nu_p(a) h(a)$, by invoking the identity (3.7) and the mean-value properties of h . The case $\alpha = 0$ is similar. Hence the relations (5.13)-(5.14) are proved. \square

Proposition 6. *A locally integrable function ϕ in D is semi-harmonic iff $\phi \in C^1(D^*)$ and there exists at each $a \in D^*$ a pseudo-ball $U \subseteq D^*$ (of radius r_a) such that*

$$(5.15) \quad \text{Res}_a(\phi, r) = 0, \quad \forall r \in (0, r_a).$$

Proof. If ϕ is semi-harmonic in D , then the relation (5.15) holds at each point $a \in D^*$ by Corollary 4.1 and the Stokes theorem. To prove the converse, assume U is a pseudo-ball at $a \in D^*$ of radius r_a satisfying the condition (5.15). Let V^j and $\hat{\phi}_j$, $1 \leq j \leq s$, be the same as in the proof of Lemma 3.1. Denoting by j_r , $r \in (0, r_a)$, the map: $\mathbb{S} = \mathbb{S}(1) \rightarrow \mathbb{S}_{[a']}(r)$, $j_r(\mathfrak{z}) = a' + r\mathfrak{z}$, one has

$$[\phi]V^j_{a,\rho} = \int_{\mathbb{S}} \hat{\phi}_j(a' + \rho\mathfrak{z}) j_{\rho}^*(\sigma_{a'}) = \int_{\mathbb{S}} \hat{\phi}_j(a' + \rho\mathfrak{z}) \sigma_0.$$

On the other hand,

$$\frac{d}{d\rho} ([\phi]V^j_{a,\rho}) \Big|_{\rho=r} = \int_{\mathbb{S}} (\partial_{\mathfrak{z}} \hat{\phi}_j)(a' + r\mathfrak{z}) \sigma_0.$$

Therefore the relations (5.4), (5.6) and (2) imply that

$$\frac{d}{d\rho} [\phi]U_{a,\rho} \Big|_{\rho=r} = \begin{cases} \left(\frac{-4\pi}{r}\right) \text{Res}_a(\phi, r), & m = 1, \\ \frac{2(1-m)}{r^{2m-1}} \text{Res}_a(\phi, r), & m > 1. \end{cases}$$

Hence $[\phi]U_{a,r} = \text{const.}$, $\forall r \in (0, r_a)$. Now the semi-harmonicity of ϕ follows from Lemma 3.3, Theorem 4.1 and Remark 1 to Theorem 4.2. \square

Another important case of an Euler type vector field associated with a smooth boundary manifold is the $\bar{\partial}$ -Neumann vector field. For the definition, let $\rho = 0$ be a local C^1 defining equation of dD in an open set $U \subseteq X^*$ with $d\rho \neq 0$ on $dD \cap U$. Define in U the $\bar{\partial}$ -Neumann vector field

$$(5.16) \quad \bar{\partial}_n := \frac{1}{\|\nabla \rho\|} \bar{\mathcal{E}}_{\rho}.$$

It follows from the relations (2.6) and (5.16) that, $\forall \phi \in C^1(\bar{U})$,

$$(5.17) \quad (\bar{\partial}_n \phi)(\zeta) = 2 \sum_{j=1}^m \rho_j(\zeta) \frac{\partial \phi}{\partial \bar{p}_j}(\zeta), \quad \zeta \in U \cap dD.$$

(For an alternative definition of the $\bar{\partial}$ -Neumann derivative, see [12], p. 62). The formula (5.17) yields a derivative of ϕ along a (complex) direction in the complex line passing through the unit outward normal to $U \cap dD$ at ζ . It is intrinsically defined. This can be seen as follows. Consider the $(m-1, m)$ -form

$$(5.18) \quad \mu_{\phi} := \sum_{k=1}^m (-1)^{m+k-1} \left(\frac{\partial \phi}{\partial \bar{p}_k} \right) dp_{[k]} \wedge d\bar{p}$$

([12], p. 2). Using the first identity in (5.8), it is easy to deduce from the definition (5.18) and the equation (5.17) the following:

Lemma 5.2.

$$(5.19) \quad (\bar{\partial}_n \phi) d\sigma_{U \cap dD} = 2^{1-m} i^{-m} j_{U \cap dD}^* \mu_{\phi}.$$

This relation implies that the definition (5.16) is independent of the choice of the local defining equation of dD . Some applications of the $\bar{\partial}$ -Neumann derivative are given in [12] and [15].

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